Solutions

DAWSON COLLEGE

DEPARTMENT OF MATHEMATICS

201-BZF-05

FINAL EXAMINATION

CALCULUS-III

Winter 2011 Time: 3 hours

Instructor: T. Kengatharam and A.Panait

Name:

ID:

Instructions:

- Translation and regular dictionaries are permitted.
- [illegible]

(1) [*5 marks*]Find the exact value of the sum

$$= \frac{e^4\pi}{\pi - e}$$

(2) [*5 marks*]Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n(x+1)^n}{(n^2+1)4^n}.$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\,(x+1)^{n+1}}{(n^2+2n+2)\,4^{n+1}} \Big/ \frac{n\,(x+1)^n}{(n^2+1)\,4^n} \right|$$

use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.)

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\int \frac{1}{1+x^2}\, dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \Rightarrow \tan^{-1} x + C = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$$

$$x=0 \Rightarrow C = 0 \Rightarrow \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$$

(5) [*5 marks*]Find the Maclaurin series of $f(x) = xe^x$ and find the radius of

(7) [*5 marks*] Find the curvature $\kappa = \frac{|\underline{r}' \times \underline{r}''|}{|\underline{r}'|^3}$ for the curve $\underline{r}(t) = (\sin(3t), \cos(3t), t)$ at the point $(0, 1, 0)$.

$$\underline{r}' = (3\cos(3t), -3\sin(3t), 1) \Rightarrow \underline{r}'(0) = (3, 0, 1)$$

$$\underline{r}'' = (-9\sin(3t), -9\cos(3t), 0) \Rightarrow \underline{r}''(0) = (0, -9, 0)$$

$$\underline{r}'(0) \times \underline{r}''(0) = \begin{vmatrix} + & - & + \\ 3 & 0 & 1 \\ 0 & -9 & 0 \end{vmatrix} = (9, 0, -27)$$

[illegible]

$$K(0) = \frac{9\sqrt{10}}{10\sqrt{10}} = \frac{9}{10}.$$

(8) [*5 marks*] Find the length of the curve $\underline{r}(t) = (2t, t^2, \frac{1}{3}t^3)$ over the interval $0 \le t \le 1$.

$$\underline{r}' = (2, 2t, t^2)$$

$$|\underline{r}'| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(t^2+2)^2} = t^2 + 2.$$

$$\text{Arc length} = \int_0^1 |r'|\, dt = \int_0^1 t^2 + 2\, dt$$

$$x^2+y^2=r^2$$

Parametrization $\underline{r}(t) = (r\cos t, r\sin t, 0)$

$\underline{r}' = (-r\sin t, r\cos t, 0)$

$\underline{r}'' = (-r\cos t, -r\sin t, 0)$

(10) [*3 marks*]Let $\underline{N}(t)$ be the unit normal vector of a space curve $\underline{r}(t)$. Prove that $\underline{N}(t)$ is orthogonal to $\underline{N}'(t)$ for all t.

$N \cdot N = 1$

$\Rightarrow N' \cdot N + N \cdot N' = 0$

$\Rightarrow 2(N' \cdot N) = 0$

$\Rightarrow N' \cdot N = 0$

$\Rightarrow N' \perp N.$

(11) [*5 marks*]Evaluate the limit if it exists or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4+y^4}.$$

along x-axis (y=0)

$f(x,0) = \dfrac{0}{2x^4+0} = 0$

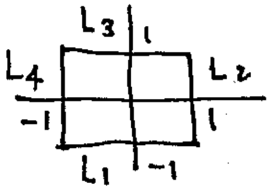
$(x,y) \to (0,0)$ along x-axis

(12) [*5 marks*]Determine the set of points on which the following function is continuous.

$$(2x, 2y, 2z) = \lambda(1, 1, 1)$$

$$2x = \lambda$$

$$2y = \lambda$$



(15) [7 marks] Find the absolute maximum and absolute minimum of

$$f(x,y) = x^2 + y^2 + x^2y + 4$$

over the rectangle $R = \{(x,y) \mid -1 \le x \le 1,\ -1 \le y \le 1\}$.

$f_x = 2x + 2xy = 0 \Rightarrow 2x(1+y) = 0 \Rightarrow x = 0$ or $y = -1$

$f_y = 2y + x^2 = 0 \Rightarrow 2y = -x^2$

$x = 0 \Rightarrow y = 0$

$y = -1 \Rightarrow x = \pm\sqrt{2}$

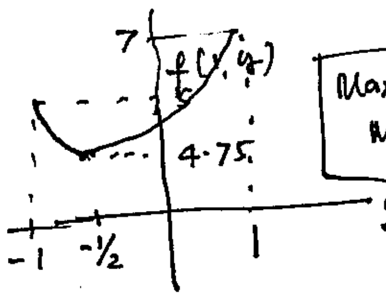
$(0,0) \in R \Rightarrow \boxed{f(0,0) = 4}$

<u>On L_1</u>: $y = -1$, $-1 \le x \le 1$

$f(x,-1) = x^2 + 1 - x^2 + 4 = 5$

<u>On L_2</u>: $x = 1$, $-1 \le y \le 1$

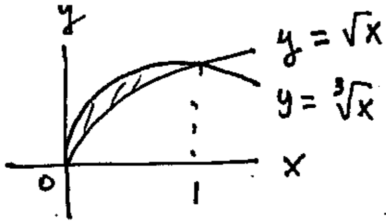
$f(1,y) = 1 + y^2 + y + 4 = y^2 + y + 5$



$\boxed{\text{Max } f(1,1) = 7 \quad \text{Min } f(1,-\tfrac{1}{2}) = 4.75}$

<u>On L_3</u>: $(y = 1,\ -1 \le x \le 1)$

$f(x,1)$



(16) [*5 marks*] Find the volume of the solid under the surface $z = 2x + y^2$ and above the region bounded by $x = y^2$ and $x = y^3$.

$$V = \int_0^1 \int_{\sqrt{x}\ \ y^3}^{\sqrt[3]{x}\ \ y^2} 2x + y^2 \, dy\, dx$$

$$= \int_0^1 2xy + \frac{y^3}{3} \Big|_{\sqrt{x}}^{\sqrt[3]{x}} dx$$

[redacted]

$$= \int_0^1 2x^{4/3} + \frac{1}{3}x - \frac{7}{3}x^{3/2} \, dx$$

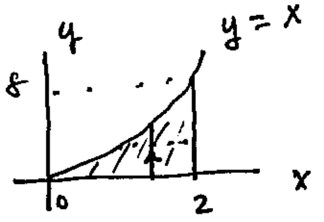
$$= \frac{2x^{\frac{7}{3}}}{7/3} + \frac{x^2}{6} - \frac{7}{3}\frac{x^{5/2}}{5/2} \Big|_0^1$$

$$= \frac{6}{7} + \frac{1}{6} - \frac{14}{15}$$

$$= \frac{19}{210}.$$

$y: 0 - 8$

$x = \sqrt[3]{y} \longrightarrow x = 2$



(17) [*5 marks*]Compute the integral

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx dy.$$

$$I = \int_0^2 \int_0^{x^3} e^{x^4} dy dx = \int_0^2 x^3 e^{x^4} dx$$

$u = x^4$

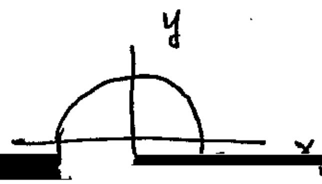
$\frac{du}{4} = x^3 dx$

$$= \frac{1}{4} \int_0^{16} e^u du.$$

$$= \frac{1}{4} e^u \Big|_0^{16}$$

$$= \frac{1}{4} (e^{16} - 1).$$

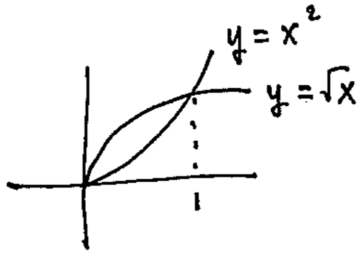
(18) [*5 marks*]Find the volume of the solid under the paraboloid $z = 18 - 2x^2 - 2y^2$ and above the xy-plane (You may use the polar coordinates).



$$V = \int^{2\pi} \int^{3} (18 - 2r^2) r dr d\theta$$

$$= \int_0 9r^2 - \frac{2r^4}{4} \Big|_0^3 d\theta = \int_0 81 - \frac{81}{2} d\theta$$

$$= \frac{81}{2} \theta \Big|_0^{2\pi} = 81\pi.$$



(19) [5 marks] Evaluate

planes $z = 0$ and $z = x + y$.

$$I = \int^{1} \int_{x^2}^{\sqrt{x}} \int^{x+y} xy \, dz\, dy\, dx = \int^{1} \int^{\sqrt{x}} x^2 y + x y^2 \, dy\, dx$$